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Nonlinear dynamics of arrays of coherent laser beams

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Nonlinear dynamics of arrays of coherent laser beams

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Summary

The final report of the project FA8655-10-1-3069 presents the combined results of the project including: introduction of the key mathematical models governing optical field propagation in the multiple-core fibres, study of the steady state propagation regimes, study of the modulation instability in multiple-core fibres, investigation of the nonlinear stage of the instability and energy transfer. We have determined theoretically thresholds of the modulation instability in the 2D infinite array of discrete waveguides, and in 2-, 3- and 4-core array systems. We have found stationary solutions in the ring configurations with a central core and N symmetrically surrounding cores and discovered in the array with non-equal cores (e.g. with the central core) the phase matching and stable steady-state propagation is possible only in the nonlinear regime. We have determined theoretically thresholds of the modulation instability in the ring configuration. The important prediction of the theory is a level of nonlinear oscillations resulting from the development of modulation instability. We studied modulation instability of such solutions in the ring configurations with a central core and N symmetrically surrounding cores. All the obtained theoretical results have been verified using numerical simulations.

1. Introduction

Recent developments in nonlinear photonics, laser physics and telecommunications attracted a great deal of attention to multi-core fibres and, in more general sense, to mathematical problems of electromagnetic field propagation in multiple interacting waveguides. In optical communications the reason behind this interest is that the transmission capacity of a single strand of fibre is fast approaching the limit (around 100 Tbit/s) set by the available optical fibre bandwidth and optical power input to the fibre. New concepts are based on the paradigm shift towards a new generation of optical infrastructure including multi-core fibre employing space division multiplexing. On the other hand, fast growing powers of modern optical devices make underlying dynamics and evolution of fields and beams essentially nonlinear. An important example is high-power fibre lasers. The fibre laser manufacturing has been greatly enhanced by the technologies developed in the telecom industry, but the recently emerged applications (and markets) include so different areas as medicine, metrology, defence, spectroscopy, industrial cutting, welding, with the list of new applications growing very fast. Fibre lasers hold a number of attractions including compactness, very good cooling characteristics and high quality of emitted light. Fibre lasers are widely considered now as promising technology of producing efficient high-power coherent light sources. However, fibre laser's brightness is limited by the variety of nonlinear effects. As the power level of fibre sources increases, nonlinear effects become increasingly important. Understanding and controlling nonlinear effects in multi-core fibre holds the key to unlocking new techniques and technologies. A number of engineering and physical techniques are used to scale fibre laser power up. Mathematical modelling plays a crucial role in building platform for new progress in this field. The way to reduce impact of detrimental nonlinear effects and to increase the peak power is to use the coherent beam combining, when the power in one beam is smaller than the threshold determined by the nonlinear processes, but the total power can be greatly higher then this threshold. Due to the fixed phase between the beams at the output of the system they can combined in one spot. Coherent combining of several fibre laser beams with a fixed phase relationship is one of the key methods to achieve high brightness sources. There are several technical approaches to coherent combining, including intra-cavity fibre couplers developed for the telecommunications industry. Another, mathematically similar promising technique is the application of a multi-core fibre as an active medium in lasers and amplifies. The key challenge in this method is the requirement of phase locking for the modes generated in different cores that are weakly coupled through fields overlapping.

Nonlinear dynamics in discrete systems is an interdisciplinary research field that has links to a large number of areas of science and technology [1-12]. This is both because matter itself is described by discrete models and also because many important engineering nonlinear systems are based on few interacting constituents or elements. Nonlinear discrete systems describe a variety of phenomena in condensed matter, nonlinear optics, biology and other fields; from energy transport in molecular chains and protein molecules to light propagation in waveguide arrays [1-12]. A number of different physical systems can be effectively described by the same mathematical models. In this report we present generic and practically very important example of low-dimension nonlinear discrete systems – light propagation in multi-core fibre and demonstrate new features introduced by discrete multi-core propagation.

2. Methods, Assumption and Procedures

We consider nonlinear dynamics of the optical field propagating in a multi-core fibre with weakly and strongly coupled cores. Figure 1 illustrates schematically the nonlinear discrete systems considered in the two previous reports here.

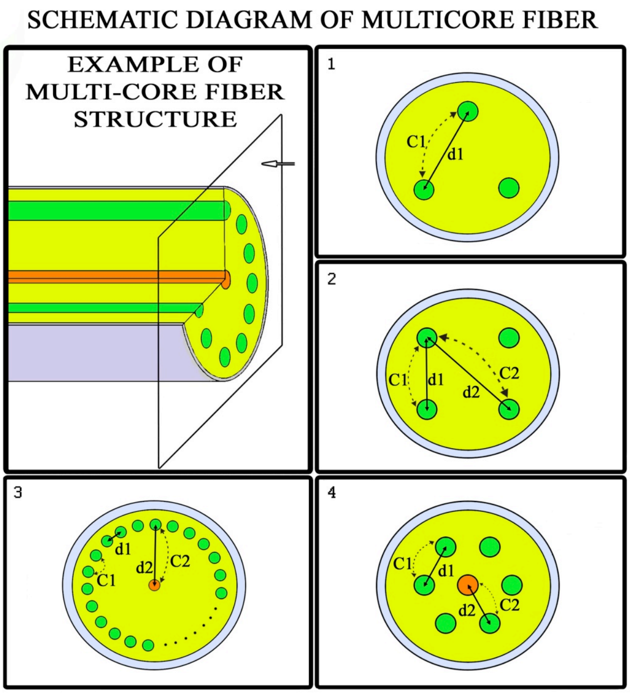

Figure 1. The schematic depiction of the multi-core fibre.

2.1 Basic equations

The basic model that we consider here is a low dimension version of the discrete nonlinear Schrödinger equation:

$$i\frac{\partial A_k}{\partial z} + \sum_{m=0}^{N} C_{km} A_m + 2\gamma_k \mid A_k \mid^2 A_k = 0,\ k = 0,\ldots N \qquad (1)$$

Here A_k a field in the k-th core, with A_0 (when applied) corresponding to the central core, $C_{mk} = C_{km}$ is the coupling coefficient between modes m and k; $C_{kk} = \beta_k$ and wave numbers in different cores and not assumed to be the same. The phase matching and stable joint CW propagation in non-symmetric arrays (e.g. 3 and 4 in Fig 1) is provided by the nonlinear phase shifts. The Eq. (1) governs all the cases shown in Fig.1:

$$(1)\ N = 2, C_{mk} = C_1;$$
$$(2)\ N = 2, C_{k,k+1} = C_1, C_{k,k+2} = C_2;$$
$$(3)\ C_{k,k\pm1} = C_1, (k \neq 0), C_{k,0} = C_2 = C_0;,$$
$$(4)\ C_{k,k\pm1} = C_1 (k \neq 0), C_{k,0} = C_2 = C_0.$$

The interaction between the cores makes the phases of radiation in different cores coupled and coherent. On the other hand, such interaction can induce the nonlinear instabilities (and periodic or irregular oscillations) which produce the intensity modulation in the cores above the destruction threshold. The instability threshold is determined not only by the laser power, but by the cores geometry, their relative positions, cladding structure and other design factors.

We study the modulation instability and determine the threshold power for specific multi-core fibre structures. We apply theoretical analysis where it is possible and also we develop specific numerical codes for the purpose of the project. The results being obtained in a very general mathematical form produce a general platform for new optical technologies of manipulation of high-power laser beams. We would like to stress that mathematical analysis of the problem is of great interest, because analytical solutions would allow designers to avoid time consuming full Maxwell equations modelling that is not very practical for design consideration. Analytical results will serve as the design guidance and as testing cases for complex numerical codes used for final pre-fabrication modelling. In the first part of the project we consider assumption of the weakly interacting waveguides that allow us to apply coupled mode theory.

3. Modulation instability in system of interacting waveguides

First recall some general properties of systems with a small number of interacting waveguides. Consider nonlinear evolution of the fields $A_k(z)$ in the multi-core fibre (k-cores) with Kerr nonlinearity. Initial condition for all cases below is: $A_k(z=0) = \sqrt{P_0}$. Due to overlap of the modes they interact linearly with some coefficient C that depends on the distance between cores and mode-field distribution. Difference in linear interactions between different modes below is geometrical. Please create all relevant pictures for each case.

3.1.1 Two-core system

First we repeat well-known results for the two core system that will be used for testing of our numerical codes.

$$i\frac{\partial A_1}{\partial z} + C\,A_2 + 2\gamma\,|A_1|^2\;A_1 = 0$$

$$i\frac{\partial A_2}{\partial z} + CA_1 + 2\gamma\,|A_2|^2\;A_2 = 0$$

Evidently, C and gamma can be scaled out of equations by scaling field A and propagation distance z. There is a well known exact analytical solution for two-core problem. However, it is illustrative to consider modulation instability for this coupler. Consider small perturbations of the CW solution of the form:

$$A_{1,2} = (\sqrt{P_0} + f_{1,2})\times\exp[i2\gamma\,P_0 z + iCz] = (\sqrt{P_0} + a_{1,2} + ib_{1,2})\times\exp[i2\gamma\,P_0 z + iCz]$$

Linearization yields the equations

$$\frac{da_{1,2}}{dz} + C\{b_{2,1} - b_{1,2}) = 0, \quad -\frac{db_{1,2}}{dz} + C\{a_{2,1} - a_{1,2}) + 4\gamma P_0 a_{1,2} = 0$$

From here $a_1 + a_2 = const, b_1 + b_2 = 4\gamma P_0(a_1 + a_2)z + b_1(0) + b_2(0)$. Assuming that $a_1 - a_2, b_1 - b_2 \propto \exp[i\Lambda z]$, (instability occurs when $\mathrm{Im}(\Lambda) < 0$), we get the dispersion relation: $\Lambda^2 = 4C[C - 2\gamma P_0]$. Instability takes place when $P_0 > C/(2\gamma)$.

For completeness of analysis we reproduce well know analytical solution for the two-core system [ ] Consider $A_1 = \sqrt{P_1}\exp[i\phi_1], A_2 = \sqrt{P_2}\exp[i\phi_2], \phi = \phi_1 - \phi_2$.

$$\frac{dP_1}{dz} = 2C\sqrt{P_1P_2}\sin(\phi) = -\frac{\partial H}{\partial \phi}$$
$$\frac{dP_2}{dz} = -2C\sqrt{P_1P_2}\sin(\phi) = \frac{\partial H}{\partial \phi}$$
$$\frac{d\phi}{dz} = 2\gamma(P_1 - P_2) + C\frac{P_2 - P_1}{\sqrt{P_1P_2}}\cos(\phi) = \frac{\partial H(P_1,\phi)}{\partial P_1} = -\frac{\partial H(P_2,\phi)}{\partial P_2}.$$

Conserved quantities (integrals of motion):

$$P_{total} = P_1 + P_2 = 2P_0,$$
$$H = 2C\sqrt{P_1P_2}\cos(\phi) - 2\gamma\, P_1P_2 = 2P_0(C - \gamma P_0) = H(P_1,\phi) =$$
$$H(P_1,\phi) = 2C\sqrt{P_1(P_{total} - P_1)}\cos(\phi) - 2\gamma\, P_1(P_{total} - P_1).$$

It is convenient to introduce:

$$\Delta P = P_1 - P_2$$

The evolution equation reads:

$$\frac{d^2\Delta P}{dz^2} = (-4C^2 + 4\gamma H + 2\gamma^2 P_{total}^2)\Delta P - 2\gamma^2\Delta P^3 \qquad (2)$$

this equation can be re-written in the dimensionless form:

$$\frac{d^2F}{dx^2} = \delta F - 2F^3, \quad x = \alpha C z, P_{cr} = \frac{C}{2\gamma}, \Delta P = 2\alpha P_{cr}F(x),$$
$$\delta = \frac{4}{\alpha^2}\left(\frac{H}{2CP_{cr}} + \frac{P_{total}^2}{8P_{cr}^2} - 1\right) = \frac{4}{\alpha^2}\Gamma$$

the solution to Eq. 2 reads:

$$\Delta P = 2\alpha\, P_{cr}\, dn(\alpha\, C(z + z_0)\,|\,k), \quad 1 < k^2 = 2 - \delta = 2 - \frac{4\Gamma}{\alpha^2} < 1.$$

Next let us examine linear instability of the solution with uniform power distribution:

$A_{1,2} = \sqrt{P_0} \times \exp[i2\gamma P_0 z + iCz]$

Consider initial conditions with small deviations of the uniform power distribution.

$$A_{1,2}(0) = \sqrt{P_0} + a_{1,2}(0) + ib_{1,2}(0), \quad a_{1,2}(0), b_{1,2}(0) << \sqrt{P_0}.$$

Then we can link initial perturbations with the initial conditions to Eq. 2:

$$\Delta P(0) = 2\sqrt{P_0}[a_1(0) - a_2(0)] = 2\sqrt{P_0}\Delta a(0),$$
$$\frac{d\Delta P}{dz}\Big|_{z=0} = 4C\sqrt{P_0}[b_1(0) - b_2(0)] = 4C\sqrt{P_0}\Delta b(0).$$

These conditions lead to relations between parameters α, k and z0:

$$dn(\alpha C z_0 \mid k) = \frac{\sqrt{P_0}\Delta a(0)}{\alpha P_{cr}},$$
$$sn(\alpha C z_0 \mid k)\, cn(\alpha C z_0 \mid k) = -\frac{2\sqrt{P_0}\Delta b(0)}{k^2\alpha^2 P_{cr}}.$$

The parameter α can be expressed through the algebraic equation:

$$\alpha^4 - 4\alpha^2\Gamma + 4\Gamma\frac{P_0\Delta a(0)^2}{P_{cr}^2} - \frac{P_0}{P_{cr}^2}\{4\Delta b(0)^2 + \frac{P_0}{P_{cr}^2}\Delta a(0)^4\} = 0$$

.

$$\alpha^2 = 2\Gamma + \sqrt{(2\Gamma - \frac{P_0\Delta a(0)^2}{P_{cr}^2})^2 + \frac{4P_0}{P_{cr}^2}\Delta b(0)^2}$$

$$k^2 = 2 - \frac{4\Gamma}{2\Gamma + \sqrt{(2\Gamma - \frac{P_0\Delta a(0)^2}{P_{cr}^2})^2 + \frac{4P_0}{P_{cr}^2}\Delta b(0)^2}} = \frac{2\sqrt{(2\Gamma - \frac{P_0\Delta a(0)^2}{P_{cr}^2})^2 + \frac{4P_0}{P_{cr}^2}\Delta b(0)^2}}{2\Gamma + \sqrt{(2\Gamma - \frac{P_0\Delta a(0)^2}{P_{cr}^2})^2 + \frac{4P_0}{P_{cr}^2}\Delta b(0)^2}} < 1$$

Thus, we can express z0 as:

$$dn(\alpha C z_0 \mid k) = \frac{\sqrt{P_0}\Delta a(0)}{\alpha P_{cr}} = dn(F(\varphi_0 \mid k) \mid k),$$

where F is elliptic integral of the first kind and

$$\sin^2\varphi_0 = \frac{1 - dn^2(\alpha C z_0 \mid k)}{k^2} = \frac{2\Gamma + \sqrt{(2\Gamma - \frac{P_0\Delta a(0)^2}{P_{cr}^2})^2 + \frac{4P_0}{P_{cr}^2}\Delta b(0)^2} - \frac{P_0^2\Delta a(0)^2}{P_{cr}^2}}{2\sqrt{(2\Gamma - \frac{P_0\Delta a(0)^2}{P_{cr}^2})^2 + \frac{4P_0}{P_{cr}^2}\Delta b(0)^2}}.$$

3.1.2 Three symmetrically placed cores

In the case of symmetrically placed three cores the key model reads:

$$i\frac{\partial A_1}{\partial z} + C\,A_2 + C\,A_3 + 2\gamma\,|\,A_1\,|^2\;A_1 = 0$$
$$i\frac{\partial A_2}{\partial z} + CA_1 + C\,A_3 + 2\gamma\,|\,A_2\,|^2\;A_2 = 0$$
$$i\frac{\partial A_3}{\partial z} + CA_1 + C\,A_2 + 2\gamma\,|\,A_3\,|^2\;A_3 = 0$$

Consider initial condition with small deviations from the uniform power distribution $A_k = \sqrt{P_0} \times \exp[i2\gamma P_0 z + i2Cz]$:

$$A_k(0) = \sqrt{P_0} + a_k(0) + ib_k(0), \quad a_k(0), b_k(0) << \sqrt{P_0}, \quad k = 1,2,3.$$

Linear evolution of the perturbations then is governed by the standard set of equations:

$$\frac{da}{dz} = M_1 b, \ -\frac{db}{dz} = M_2 a, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \ b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 2C & -C & -C \\ -C & 2C & -C \\ -C & -C & 2C \end{pmatrix}, M_2 = \begin{pmatrix} -4\gamma P_0 + 2C & -C & -C \\ -C & -4\gamma P_0 + 2C & -C \\ -C & -C & -4\gamma P_0 + 2C \end{pmatrix}.$$

Considering $a, b = a(0), b(0)e^{i\Lambda z}$ we get determinant $\det(M_1 M_2 - \Lambda^2 I) = 0$

$$\Lambda_{1,2} = 0, \ \Lambda_{3,4,5,6} = \pm\sqrt{3C(3C - 4\gamma P_0)}$$

Instability occurs when $P_0 > \dfrac{3C}{4\gamma}$.

3.1.3 Four cores in a square lattice

In the case of the four cores there are two main options: square lattice and one core in the centre. We consider first square lattice case.

$$i\frac{\partial A_1}{\partial z} + C_1\, A_2 + C_2\, A_3 + C_1\, A_4 + 2\gamma\, |\, A_1\,|^2\, A_1 = 0$$

$$i\frac{\partial A_2}{\partial z} + C_1 A_1 + C_1\, A_3 + C_2\, A_4 + 2\gamma\, |\, A_2\,|^2\, A_2 = 0$$

$$i\frac{\partial A_3}{\partial z} + C_2 A_1 + C_1\, A_2 + C_1\, A_4 + 2\gamma\, |\, A_3\,|^2\, A_{23} = 0$$

$$i\frac{\partial A_4}{\partial z} + C_1 A_1 + C_2\, A_2 + C_1\, A_3 + 2\gamma\, |\, A_4\,|^2\, A_4 = 0$$

Here C_d stands for diagonal coupling between modes. It is evident that $C_2 < C_1$.

Similar to previous cases, consider initial condition with small deviations from the uniform power distribution $A_k(z) = \sqrt{P_0} \times \exp[2i\gamma P_0 z + 2iC_1 z + iC_2 z]$:

$$A_k(0) = \sqrt{P_0} + a_k(0) + ib_k(0), \quad a_k(0), b_k(0) << \sqrt{P_0}, \quad k = 1,2,3,4.$$

Instability occurs when $P_0 > \dfrac{C_1 + C_2}{2\gamma}$.

3.2 Modulation instability in the infinite quadratic lattice

We consider now the case of the infinite array of weakly interaction nonlinear waveguides as a reference point for future optimisation of the design of arrays. In the case of the infinite square array, the master equation reads

$$i\frac{\partial A_{n,m}}{\partial z} + C\,(A_{n,m+1} + A_{n,m-1} + A_{n+1,m} + A_{n-1,m})$$
$$+ 2\gamma\, |\, A_{n,m}\,|^2\, A_{n,m} = 0$$

Consider modulation instability for this lattice with the small perturbations of the CW in the following form:

$$A_{nm} = (\sqrt{P_0} + f_{n,m}) \times \exp[i2\gamma P_0 z + i4Cz] = (\sqrt{P_0} + a_{n,m} + ib_{n,m}) \times \exp[i2\gamma P_0 z + i4Cz]$$

Linearization yields the equations

$$\frac{da_{n,m}}{dz} + C\{b_{n+1,m} + b_{n-1,m} + b_{n,m+1} + b_{n,m-1} - 4b_{n,m}) = 0$$

$$-\frac{db_{n,m}}{dz} + C\{a_{n+1,m} + a_{n-1,m} + a_{n,m+1} + a_{n,m-1} - 4a_{n,m}) + 4\gamma P_0 a_{n,m} = 0$$

Assuming that $a_{n,m}, b_{n,m} \propto \exp[i\Lambda z + ik_1 n + ik_2 m]$,
(instability occurs when $\mathrm{Im}(\Lambda) < 0$), we get the dispersion relation:

$$\Lambda^2 = 4C[\cos(k_1) + \cos(k_2) - 2][C\{\cos(k_1) + \cos(k_2) - 2\} + 2\gamma P_0]$$

Instability occurs when:

$$[\cos(k_1) + \cos(k_2) - 2][C\{\cos(k_1) + \cos(k_2) - 2\} + 2\gamma P_0] < 0$$

The instability occurs when power is larger than a critical value

$$2\gamma P_0 > C\{1 - \cos(k_1) + 1 - \cos(k_2)\} = 2C\{\sin^2(\frac{k_1}{2}) + \sin^2(\frac{k_2}{2})\}$$

In the limit small k it gives criterion of the instability:

$$P_0 > \frac{C}{\gamma}\{\frac{k_1^2}{2} + \frac{k_2^2}{2}\}$$

4. Non-symmetric multi-core systems

4.1 Mathematical model

The case of multiple peripheral cores and one in the center the general solution is not straightforward. However, in many situations dynamics in mutli-core systems can be reduced (assuming $A_k = A_1, k = 1,...,N$) to analysis of effective two-core model that can serve as a symmetric limit to multi-core systems:

$$i\frac{\partial U_0}{\partial z} = -U_1 - \frac{2N\gamma_0}{\gamma_1}\left|U_0\right|^2 U_0 = \frac{\partial H}{\partial U_0^*}, \qquad (3)$$

$$i\frac{\partial U_1}{\partial z} = -\kappa U_1 - U_0 - 2\left|U_1\right|^2 U_1 = \frac{\partial H}{\partial U_1^*}. \qquad (3)$$

Here normalised functions and variables are introduced:

$$A_{0,1} = \sqrt{P_{0,1}}U_{0,1}e^{i\beta_0 Lz};\ z' = z / L;\ L = 1 / C_0\sqrt{N},$$

$$P_0 = NP_1 = N^{3/2}C_0 / \gamma_1,\ \kappa = \frac{(\beta_1 - \beta_0) + 2C_1}{C_0\sqrt{N}}.$$

The system (3) is a Hamiltonian one with the following conserved quantities:

$$P_{total} = N(|U_0|^2 + |U_1|^2),$$

$$H = -\kappa |U_1|^2 - (U_0^* U_1 + U_1^* U_0) - |U_1|^4 - \frac{N\gamma_0}{\gamma_1}|U_0|^4 .$$

4.2 Steady state solutions

We would like to stress that despite simple appearance, even the stationary, steady state solution of the system (3) is non-trivial anymore (compared e.g. to the symmetric dimer/coupler). To provide for coherent light evolution in multiple cores, difference in propagation constants has to be compensated by the nonlinear phase shifts. Consider steady-state solutions of the system (3) in the form:

$$\{U_0, U_1\} = \{A, B\} \times \exp[i\lambda z],$$

$$\Gamma = \frac{B}{A}, \quad |A|^2 = \frac{P_{total}}{N(1+\Gamma^2)}, \lambda = \Gamma + \frac{2\gamma_0 P_{total}}{\gamma_0(1+\Gamma^2)}.$$

$$\Gamma^4 - (\kappa + \frac{2P_{total}}{N})\Gamma^3 - (\kappa - \frac{2\gamma_0 P_{total}}{\gamma_1})\Gamma - 1 = 0.$$

The relatively simple mathematical result leads to quite nontrivial physical consequences. Namely, steady state dynamics in such system is possible only with a certain imbalance (given by factor Γ^2) between powers propagating in different cores. This imbalance is due to nonlinear contribution to the phase matching condition of the propagation constants. The physics is rather transparent - this power split is due to nonlinear phase shift contribution to the phase matching condition required for coherent propagation in multiple cores. Note that there are several power distributions (between central and peripheral cores) that can provide for a coherent steady state propagation of light. The amount of power that has to be coupled to each core for steady state evolution given by solutions above depends on four parameters: (i) number of surrounding cores *N*, (ii) input power *P_in* (or total power *P_total*), (iii) linear phase mismatch kappa, and (iv) the ratio between the nonlinear coefficients. Example of the four families of steady state solutions are presented in Figs. 2, 3.

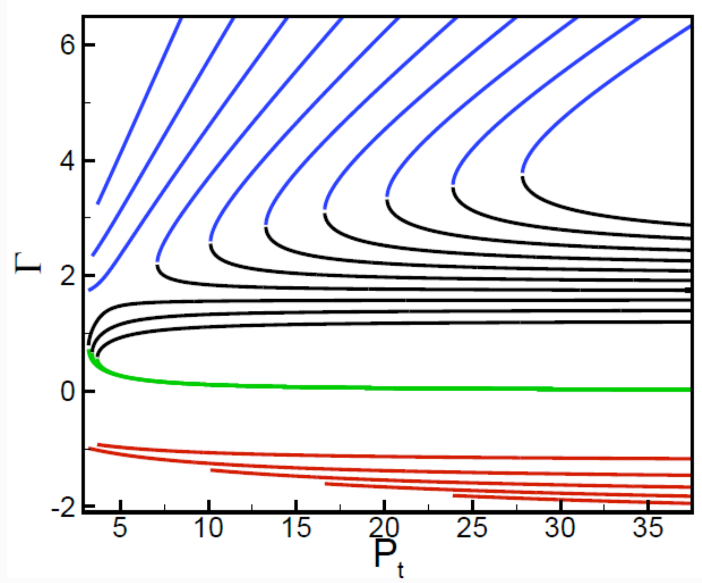

Figure 2. Four values of Γ corresponding to different power splits between cores as functions of total input power; here $\gamma_0/\gamma_1 = 0.5, \kappa = 1$. Blue, green and read branches are stable while the black one is unstable. Here different curves for each branch correspond to *N* varying from 3 to 12 (from the bottom to the top). For red curve only odd N are shown.

Green line in Figs. 2, 3 corresponds to propagation of the most of power in the central core. Negative Γ means out-of phase fields in the central and peripheral cores. Note that when light is launched into the central core power will be coupled to the peripheral cores in the ring. The amount of power that will be coupled to each core depends on N, input power, phase mismatch and ratio between nonlinear coefficients.

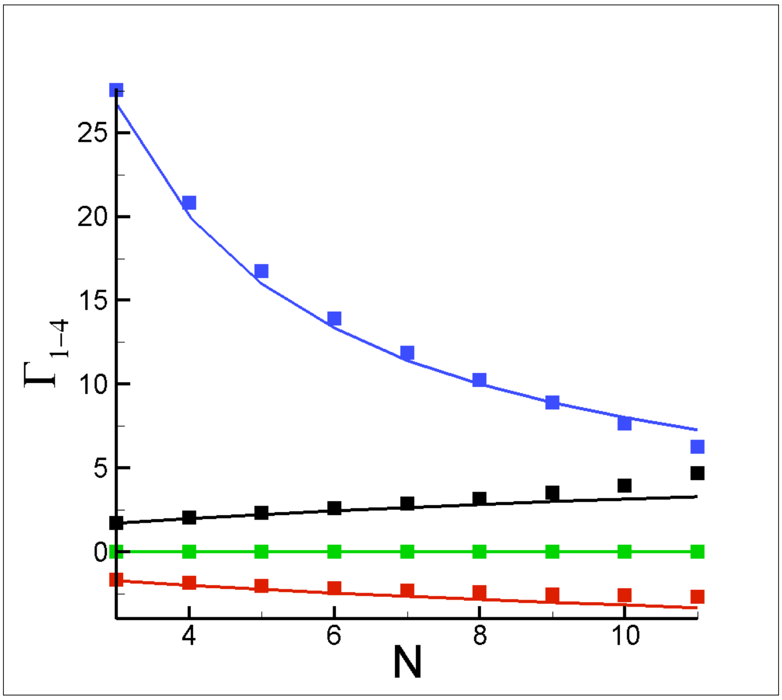

Figure 3. Dependence of the four solutions of Eq. (10) (shown by squares) on N. Here $\gamma_0/\gamma_1 = 0.5, \kappa = 1, P_t = 40$. Solid lines are for the following analytical asymptotic solutions valid in the limit P_t ≫ 1.

Blue curve: $\Gamma_1 \propto 2P_t/N$;

Black line: $\Gamma_2 \propto \sqrt{\gamma_1 N/\gamma_0}$;

green line: $\Gamma_4 \propto \dfrac{\gamma_t}{2\gamma_0 P_t}$

and red line: $\Gamma_3 \propto -\sqrt{\gamma_1 N/\gamma_0}$

It is seen that analytical approximation show very good agreement with numerical modelling in this limit.

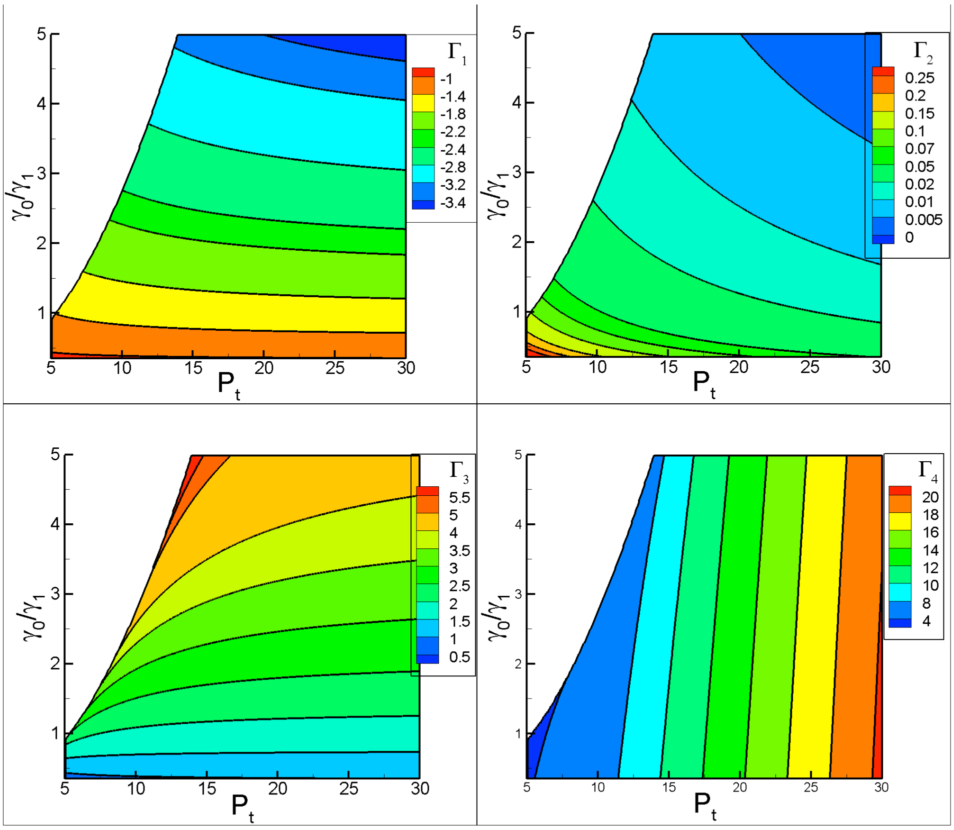

Figure 3B. Four values of Γ corresponding to different power splits between cores as functions of total input power P_t and γ_0/γ_1. Here $\kappa = 1$, N=3.

4.3 Stability of the steady state solutions

Consider now stability of steady state solutions of - analogue of the modulation instability for considered low dimension discrete system (3). The small amplitude disturbance is taken in a standard form:

$$\{U_0, U_1\} = \{A + a + ib, B + c + id\} \times \exp[i\lambda z],$$

for perturbations proportional to $\exp[pz]$ the increment of instability is given (omitting details of calculations) by the following expression:

$$p^2 = -(2 + (\frac{1}{\Gamma} - 4B^2)\frac{1}{\Gamma} + (\Gamma - 4\frac{N\gamma_0}{\gamma_1}A^2)\Gamma)$$

Instability results in periodic oscillations of energy between cores with amplitude of modulations depending on total power, i.e. the relative modulation depth decreases with growing input power. The most important consequences of the instability is that it makes control of power dynamics hardly possible. For system with more than three cores the instability, in general, produces stochastic modulation breaking the mutual coherence in the cores. The energy exchange oscillations can be produced not only as a result of the *instability*, but also as a result of *initial conditions* (in case of arbitrary input powers).

4.4 Energy exchange in the case of uniform initial power distribution

The Hamiltonian structure of the equations (3) and the additional conserved quantity greatly restricts dynamics in the considered low dimension dynamic system imposing constraints on the evolution of the waves and the energy exchange between cores. For instance, consider evolution of input condition with power initially equally distributed between all cores. Energy exchange can be characterized by the following function:

$$\Delta U_0 = (N |U_0|^2 - |U_1|^2)/P_t$$

Now using restrictions imposed by the Hamiltonian it is easy to show that the complete energy transfer from the outer cores to the central one is possible only for one specific value of input power (and at specific propagation length):

$$P_{in} = P_{in}^{th} = \frac{\kappa + 2N^{-1/2}}{\gamma_0 (N+2)/\gamma_1 - 1}$$

The observed effect - localization of all initially evenly distributed power into the central core can be considered as an ultimate discrete version of the self-focusing of light. The analytical expression above describing condition of the complete energy exchange, in particular, gives an estimate of the number of cores required to convert all power e.g. from the silica-based peripheral cores to the central air-core in a hollow-core photonic crystal fibre. Figure 4 shows comparison of the analytical result and numerically calculated threshold of a complete energy transfer given by

$$\Delta U_0 = (N |U_0|^2 - |U_1|^2)/P_t$$

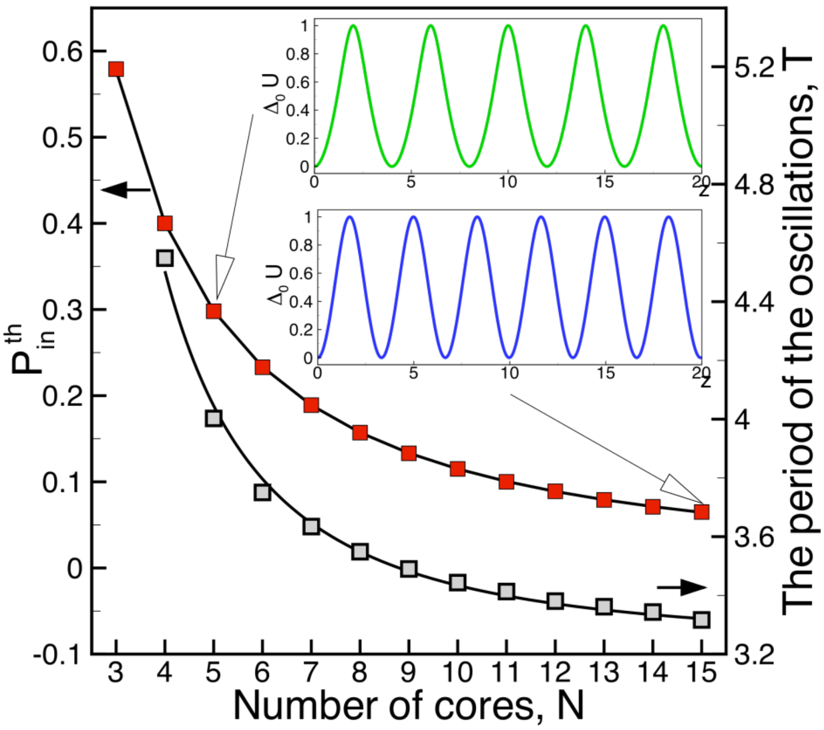

Figure 4. X-axis: energy transfer (red markers) and the analytical formula (solid line). Y-axis to the right: numerically calculated period of the power oscillations (gray markers) and analytical approximation: $3.23 + 2.04/N^2$(solid line). Insets show complete energy transfer at certain distances. Here $\gamma_0 = \gamma_1; C_1 = C_0; \beta_1 = \beta_2$. The period of the energy exchanges decays with *N* as N^2.

Note that the presented theory can be easily generalized to pulse propagation and nonlinear temporal dynamics having numerous applications. In the recent important work [13] it has been studied the efficiency of nonlinear matching of optical fibres through a fundamental soliton coupling from one fibre into another opening a range of engineering applications, e.g. optimized Raman red-shift and supercontinuum generation.

4.4 Instability in the case of angular perturbations

Now we consider angular instabilities in the system with one central core N and peripheral. It is convenient to re-write equations in the following dimensionless form:

$$i\frac{\partial U_0}{\partial z} = -\frac{1}{N}\sum_j U_j - 2\frac{\gamma_0}{\gamma_1}N|U_0|^2 U_0$$

$$i\frac{\partial U_j}{\partial z} = -\kappa U_j - \frac{C_1}{C_0\sqrt{N}}\left(U_{j+1} + U_{j-1} - 2U_j\right) - U_0 - 2N|U_j|^2 U_j$$

Consider now perturbations of general form:

$$U_0 = e^{i\lambda z}(A + a + ib)$$

$$U_j = e^{i\lambda z}(B_j + (f + ig) + (c + id)e^{isj}); a,b,c,d,f,g \propto e^{pz}$$

Resulting spectral problem for the angular perturbation equations reads:

$$-pd + (\kappa - \lambda + 6NB^2)c + 2\frac{C_1}{C_0\sqrt{N}}(\cos s - 1)c = 0$$

$$pc + (\kappa - \lambda - 2NB^2)d + 2\frac{C_1}{C_0\sqrt{N}}(\cos s - 1)d = 0$$

It is straightforward to derive the instability increment:

$$\left(-\lambda + k + 2NB^2\right) = -\frac{A}{B}; C = \frac{C_1}{C_0\sqrt{N}}$$

$$p^2 = \left(\frac{A}{B} + 4C\,Sin^2[\frac{s}{2}]\right)\left(4NB^2 - \frac{A}{B} - 4C\,Sin^2[\frac{s}{2}]\right)$$

5. Stochastization above instability threshold

Next we present an example of numerical simulation of modulation instability in the square lattice system with four cores. In the general case power transfer cannot be fully described by analysis of the Hamiltonian and numerical modelling is required. Consider nonlinear evolution of the fields $A(z)_k$ in the fouri-core fibre (k-cores) with Kerr nonlinearity. Consider initial condition with small deviations from the uniform power distribution:

$$A_k(0) = \sqrt{P_0} + a_k(0) + ib_k(0), \quad a_k(0), b_k(0) << \sqrt{P_0}, \quad k = 1,2,3,4.$$

We present in this subsection results for different initial power P_0 and

$$a_1 = a_3 = -10^{-3}, a_2 = a_4 = 10^{-3}, b_1 = b_2 = b_3 = b_4 = 0.$$

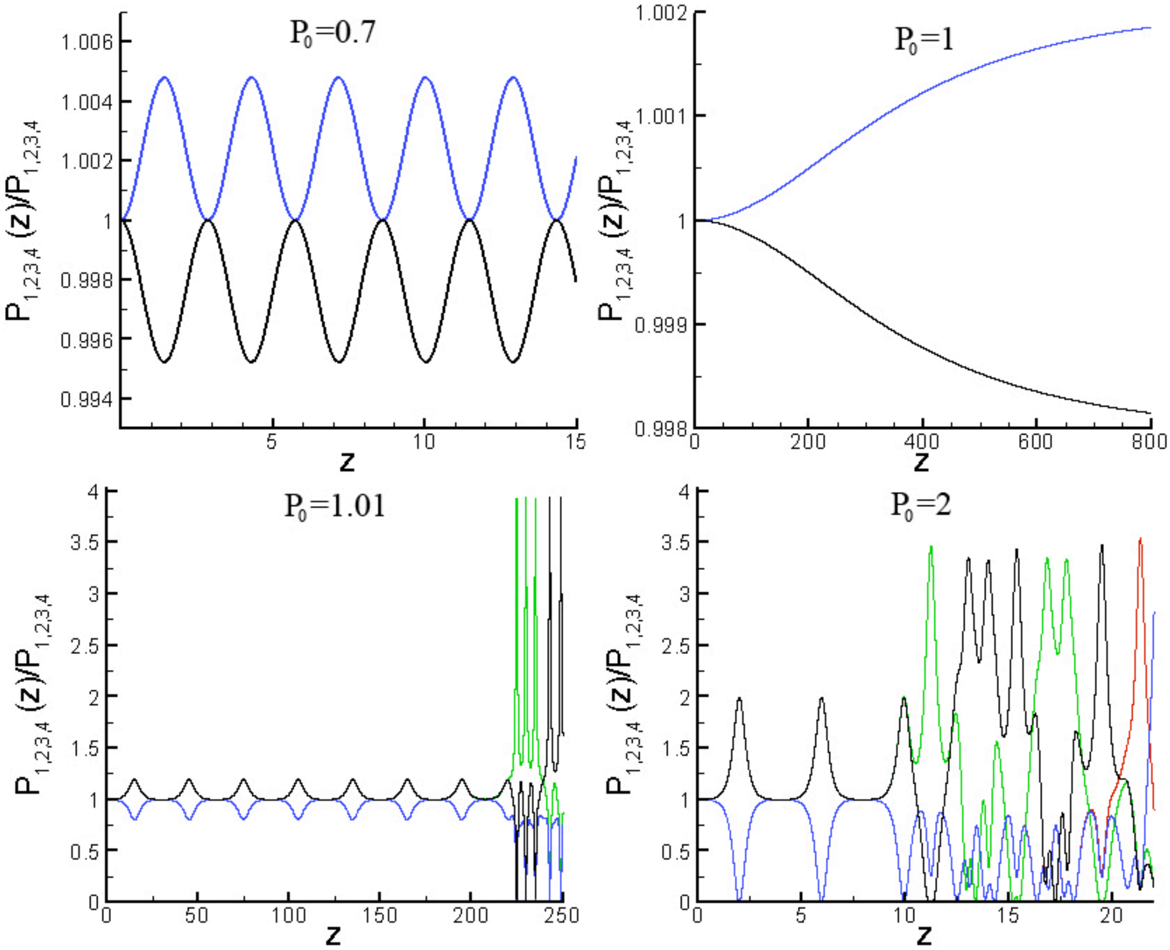

Figure 5. Power evolution in first core (red line), the second core (green line), third core (blue line) and fourth core (black line) for different input powers P_0.

Figure 5 shows that transition to instability in the case of four cores easily leads to stochastisation. Figure 5 (top left) shows stable dynamics below threshold (equal to 1). Figure 5 (top right) presents development of instability in the case of initial power equal to the threshold and only minimally exceeding the threshold because of perturbations. In this case one can see monotonic power exchange over very long distance. However, a small increase of input power (Fig. 5, bottom left) dramatically changes the power evolution. As it can be seen from Fig. 5 the dynamics becomes very irregular after certain distance. In the case of multiple cores, the modulation instability can lead to fats stochastisation and non-predictable evolution of powers that might lead to engineering challenges in multi-core systems.

Conclusions

In this project we studied mathematical models describing propagation of optical field in multi-core nonlinear systems. We introduced models governing optical field propagation in the multiple-core fibres. We have determined theoretically thresholds of the modulation instability in the 2D infinite array of discrete waveguides, and in 2-, 3- and 4-core array systems. The obtained theoretical results have been verified using numerical simulations. We have analysed the instabilities in the ring configurations with a central core and N symmetrically surrounding cores. This system can be analysed by reduction to an effective generalised non-symmetric two-core system. The non-symmetric two-core system has nontrivial steady state solutions with non-equal power distribution between the cores. The balance can be achieved only with corresponding contribution of the nonlinearity. We have found stationary solutions based on a balance provided by nonlinear phase shift. We have next determined theoretically thresholds of the modulation instability in the ring configuration. The obtained theoretical results have been verified using numerical simulations. The important prediction of the theory is a level of nonlinear oscillations resulting from the development of modulation instability. We have started analysis of stochastisation in four- and more core systems.

In general, we have presented a theory energy transfer in low dimension arrays of coupled nonlinear waveguides. The developed theory is rather generic and has a range of potential applications. Without loss of generality, particular emphasis in the analysis is made on multi-core fibre technology, important in the fields of both high power fibre lasers and ultra-high-capacity optical communication systems. We have derived for the array with non-equal cores the nonlinear phase matching conditions that provide for stable coherent steady-state propagation in multiple cores. We solved the stability problem and found an exact analytical condition of complete energy transfer from peripheral to the central core - ultimate discrete analogy of the self-focusing effect.

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